Gödel's Incompleteness Theorem

Overview

Computability and Logic

Recap

- Remember what we set out to do in this course: Trying to find a systematic method (algorithm, procedure) which we can use to decide, for any statement about mathematics, whether that statement is true or false.

- In short: Is there a decision procedure for mathematical truth?

Our Formal Logic-Based Attempt

- Our initial attempt was based on formal logic:
  1. Use FOL to symbolize statements about mathematics
  2. Declare a subset of these statements as axioms: statements that we know to be true
  3. Try to decide the truth of any mathematical statement by deciding whether or not it is a logical consequence of the axioms

Formal Proofs

- For step 3, we contemplated the use of formal proofs.
- That is, maybe we can rephrase the question:

  "Is statement S a logical consequence of axiom set A?"

  with:

  "Can statement S be formally derived from axiom set A?"

Some Immediate Issues

- Formal proofs demonstrate consequence, but not non-consequence
- Formal proof systems themselves aren't systematic

- But maybe a systematic method can nevertheless be created on the basis of formal logic?
  - Truth trees are systematic … and can demonstrate consequence as well as non-consequence. Cool!
    - … but sometimes trees get infinitely long. Not cool!
  - Is there some other procedure? … not sure. Let's set this question aside.

Peano Axioms

- We tried a very small set of 6 axioms, called the Peano axioms, designed for a small subset of mathematics: natural number arithmetic.
- We found that we could indeed prove several (non-trivial) theorems about arithmetic from the Peano Axioms. Cool!
- We also found that some arithmetical truths could not be derived from the original 6 Peano axioms. Not cool!
- But then we also found that if we added an axiom scheme reflecting mathematical induction, we could prove many more arithmetical truths. Cool!
- Can all arithmetical truths be derived from this set? In other words, is PA complete? … Not sure. Let’s set this question aside for a bit as well.

Gödel's Completeness Result: FOL is Complete!

- In 1929, Gödel showed that for any axiom set A and statement S, if S is a logical consequence of A, then there exists a formal proof that derives S from A.
- Cool! So yes, we can replace the question about consequence with a question about provability.
- Now we just need a procedure that eventually:
  1. Derives S from A if S follows from A
  2. Concludes that S cannot be derived from A if S does not follow from A
- Let's go back to the question "Is PA (or some other axiom set) complete for arithmetic?"

Expressive Completeness

- Notice that PA uses L_A = {0, s, +, *} as its only non-logical symbols.
- Can all arithmetical statements be expressed using this very restricted set of symbols?
- How, for example, would you even express the Fundamental Theorem of Arithmetic (every number has a unique prime factorization)? Is that even possible?
- Again, we'll set this question aside for now.
- For now, we can contemplate a notion of completeness relative to our language L_A:
  - Axiom set A is complete iff for all S□ L_A : if S is true (i.e. if for standard interpretation N: N ⊨ S), then A ⊨ S.

A Trivially Complete Axiom Set

- Consider A = $\{\bot\}$.
  - Clearly, A is complete: all arithmetical truths can be derived from A! Cool!
  - But: all arithmetical falsehoods can be derived from A as well! Not cool! … Very not cool!
  - OK, so any axiom set we want should be *sound*: all statements that follow from it should be true.

Another Trivially Complete Axiom Set

- Consider $A = \{S \in L_A \mid N \models S\}$
  - Again, clearly, A is complete. Cool!
  - OK, but this isn't what we would intuitively consider an 'axiom set': it goes against the whole idea of deriving all theorems from a small set of basic and elementary truths. Not cool!
  - More importantly, we can't work with this as part of any effective procedure. We don't know what the axioms are. We don't have an 'effective' starting point. Very not cool!
  - So, any axiom set should be such that for any statement, we can effectively decide whether or not it is an axiom.
    - Notice that this *does* allow the inductive axiom scheme, representing an infinite number of axioms, as part of the axiom set
  - By the Church-Turing Thesis, deciding whether some object is an element of some set amounts to that S being recursive.
  - So, is there a sound and recursive axiom set A that is complete?

Gödel's Incompleteness Result (1931): Arithmetic is Incomplete

- In 1931, the bomb dropped: Kurt Gödel proved that *There is no complete (sound and recursive) axiom set for natural number arithmetic.*

- Gödel's Incompleteness Theorem is regarded as one of the most important theorems of the 20th century!

The Liar Paradox

- Consider the following statement P:
  - This statement is false
- If P is true, then P is false, and if P is false, then P is true. Contradiction!
- OK, so what does this mean?
  - That not every statement is true or false?
  - That only meaningful statements are true or false, but statement is not meaningful?
  - Nobody really knows how to think about this or how to resolve the paradox!
- Gödel's proof feels very much like the Liar Paradox!

Gödel Numbering

- Key to Gödel's proof was his Gödel numbering: using numbers to encode FOL symbols, expressions, proof structures, and other kinds of syntactical FOL objects.
- The encoding is effective: given an FOL object, there is an effective procedure to encode that object.
- The decoding is effective too: given an encoding of some FOL object, one can effectively figure out what object is being encoded.

Definability

- Gödel next showed that various kinds of properties, relations, and functions regarding natural numbers (in particular, those that are relevant to the Gödel encodings) can be expressed ('defined') by FOL statements using the language of arithmetic {0, s, +, *} as its non-logical symbols.
- E.g. 'primeness' is definable since there is an FOL expression using {0, s, +, ×} as its only non-logical symbols that 'captures' primeness:

  x is prime iff $\exists z\ s(s(0) + z) = x \wedge \forall y\ (\exists z\ s(y + z) = x \rightarrow (\exists z\ y \times z = x \rightarrow (y = s(0) \vee y = x)))$

Self-Reference

- Since (some) numbers represent FOL objects according to the Gödel numbering, FOL statements about numbers can be used to make statements about … FOL statements (and other FOL objects)!

Coding Syntactical Properties

- E.g. We can define a formula “Sentence(x)” which will be true iff x is the Gödel number of a FOL sentence. In other words, ‘sentence-ness’ is definable (in L_A).

- You can also show that for any recursive set of axioms A (expressed in L_A), there is a definable (in L_A) expression Axiom(x) such that Axiom(**n**) is true iff n is the Gödel number of some axiom in A. (remember: **n** = s(s(…s(0) ..)) (n times))

Inference Relationships

- An especially important syntactical claim about statements in L_A is the inferential relationship.
- Gödel showed that given some syntactical inference rule R you can define an expression $Derivable_R(x,y)$ that states that y is the code of some sentence that can be syntactically derived by a set of sentences encoded by x.
- So, if you take some sound and complete system S of inference (which Gödel showed exists), you can define Implies(x,y) that states that y is the code of some sentence that is logically implied by a set of sentences encoded by x.

Proof Properties

- Gödel then showed that for any recursive set of axioms A, there is a definable expression Proof(x,y) such that Proof(**n**,**m**) is true iff n is the code of a proof whose premises are members of A and whose conclusion is a sentence whose code is m (for this you use the Axiom(x) and Implies(x,y) expressions).
- This means that there is also a formula Provable(x) = $\exists$y Proof(y,x) that defines the property of being provable from A.
- So: $\neg$Provable(x) defines "unprovability (from A)"!

The Diagonal Lemma

- A final key step in Godel’s proof was to prove the Diagonal Lemma:
- For any wff A(x) there exists a sentence G such that G is logically equivalent to A(**g**), where g is the Gödel number of G.
- In other words, for any formula (property) A(x), there is a sentence that says “I have property A(x)”

Gödel Sentences

- By the Diagonal Lemma, for any recursive and sound set of axioms A, there exists a sentence G such that G is equivalent to $\neg$Provable(g) where g is the Gödel number of G.
- This G is called the “Gödel sentence”, which basically says “I am not provable (from A)”.
- Now, if G_A is false, then it can be proven from A. But that would mean that A is not sound. Since A is sound, that means that G_A is true. So it is true that G_A is not derivable from A. So, there is a true statement that cannot be derived from A: A is incomplete!

End

Part III - Representability

- In fact, Gödel showed that various statements about these properties are logical consequences of ('represented by') the 6 Peano Axioms.
- E.g. for any prime number n, the FOL expression "Prime(n)" is a consequence of PA1-6. That is, where **n** = s(s(…s(0)…)) (n times):
  - i.e. $\exists z\ s(s(0) + z) = \mathbf{n} \wedge \forall y\ (\exists z\ s(y + z) = \mathbf{n} \rightarrow \exists z\ (y * z = \mathbf{n} \rightarrow (y = s(0) \vee y = \mathbf{n})))$ can be derived from PA1-6

Coding Syntactical Properties

- E.g. We can define a formula “Sentence(x)” which will be true iff x is the Gödel number of a FOL sentence, and we can show that if n is the Gödel number of a sentence, then “Sentence(**n**)” can be derived from PA1-6. In other words, ‘sentence-ness’ is definable (in L_A) and representable (in PA).

- You can also show that for any recursive set of axioms A (expressed in L_A) that is at least as strong as PA, there is a definable (in L_A) and representable (in A) expression Axiom(x) such that Axiom(**n**) is true iff n is the Gödel number of some axiom in A.

Sketch of Proof of Diagonal Lemma I

- The diagonalization of an expression A(x) (of L_A) is the expression $\exists x\ (x = \mathbf{a} \wedge A(x))$, where a is the Gödel number of A.
- There is a formula Diag(x,y) such that Diag(**m**,**n**) is true iff n is the Gödel number of the diagonalization of the expression whose Gödel number is m.

Sketch of Proof of Diagonal Lemma II

- Let A(x) be the formula $\exists y\ (\mathrm{Diag}(x, y) \wedge B(y))$, with Gödel number a.
- Let G be the diagonalization of A(x), i.e. G is the sentence $\exists x\ (x = \mathbf{a} \wedge \exists y\ (\mathrm{Diag}(x, y) \wedge B(y)))$
- So G basically says: “The diagonalization of A(x) has property B”.
- But since the diagonalization of A(x) is G itself, G ends up saying “I have property B”